

# The scalar product of XXZ spin chain revisited. Application to the ground state at $\Delta = -1/2$

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## Abstract

For the scalar product  $S_n$  of the XXZ  $s = 1/2$  spin chain we derive a new determinant expression which is symmetric in the Bethe roots. We consider an application of this formula to the inhomogeneous groundstate of the model with  $\Delta = -1/2$  with twisted periodic boundary conditions. At this point the ground state eigenvalue  $\tau_n$  of the transfer matrix is known (see e.g.[7]) and has a simple form that does not contain the Bethe roots. We use the knowledge of  $\tau_n(\mu)$  to obtain a closed expression for the scalar product. The result is written in terms of Schur functions. The computations of the normalization of the ground state and the expectation value of  $\sigma^z$  are also presented.

## 1 Introduction

The computation of the correlation functions of the integrable XXZ spin-1/2 chain of finite length  $N$  can be done using the form factor approach [10, 9]. These are the form factors of the local spin operators, they admit a nice determinant representation based on the Slavnov determinant [18, 19]. The Slavnov determinant is a scalar product of two states in the Algebraic Bethe Ansatz picture. Each of these states depend on the corresponding sets of rapidities of which one set is taken free and the other one is the set of Bethe roots. Thus, it is a scalar product between a Bethe state and an off-shell state. The Bethe roots satisfy a system of nonlinear algebraic equations (the Bethe equations). At this point, to obtain an explicit answer for the scalar products one must know the solution of the Bethe equations. In general, analytical computations for finite systems stop here. In the case when the interaction parameter  $\Delta = -1/2$  (or  $q^3 = 1$ , where  $q$  is the deformation parameter of the  $U_q(A_1^{(1)})$  quantum group), called the combinatorial point of the model, it is possible to obtain a formula for the ground state scalar product which has no Bethe roots dependence. The answer is given in terms of a Schur function and its derivation is the purpose of the present work.

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The twisted XXZ spin-1/2 chain at  $\Delta = -1/2$  attracted a lot of attention in the past fifteen years due to the relation of its ground state with other statistical models. The numbers appeared in the ground state eigenvector of the transfer matrix are related to the combinatorics of the alternating sign matrices [1, 17]. A spin chain state can be mapped to the so called loop basis, relating the XXZ to the dense Temperley-Lieb (TL) loop model. In the TL model the condition  $\Delta = -1/2$  is translated to the fact that the loop weight  $n$  is equal to 1. In this case this loop model describes interesting statistical systems like the critical bond percolation. A number of works [15, 14, 6, 4, 8] were devoted to study correlation functions of this model. Due to the relation between the ground state of the spin chain and the ground state of the loop model the knowledge of the form factors on the spin side may serve as a tool for the computation of some correlation functions in the loop model.

The technical overview of our work is the following. Our derivation of the ground state scalar product at  $\Delta = -1/2$  and twist  $\kappa = q^2$  is based on a few important aspects. First, we rewrite the scalar product in a symmetrized form. In the original determinant expression for the scalar product of an  $n$ -particle state:

$$S_n \propto \det_{1 \leq i, j \leq n} (s_{i,j}), \quad (1)$$

each matrix element  $s_{i,j}$  is a function of the Bethe roots, which will be called  $\zeta_1, \dots, \zeta_n$ , a set of free parameters, called  $\mu_1, \dots, \mu_n$  and a set of inhomogeneities  $z_1, \dots, z_N$ . The functions  $s_{i,j}$  can be written as derivatives of the  $n$ -particle eigenvalue of the transfer matrix  $\tau_n(\mu_i | \zeta_1, \dots, \zeta_n)$  with respect to the Bethe roots:

$$s_{i,j} = \frac{\partial \tau_n(\mu_j | \zeta_1, \dots, \zeta_n)}{\partial \zeta_i}. \quad (2)$$

The functions  $s_{i,j}$  are not symmetric in the Bethe roots in the original determinant expression [18, 19] and after taking the determinant we get extra factors of a Vandermonde determinant of  $\zeta$ 's and of  $\mu$ 's. If  $s_{i,j}$  were symmetric in  $\zeta$ 's we could simply rewrite them in terms of symmetric combinations of the Bethe roots, say, in terms of the elementary symmetric polynomials. Then, using the knowledge of the  $Q$ -function (computed in [7]) for our special case, we could eliminate the elementary symmetric polynomials of the Bethe roots and obtain the desired formula. However, the functions  $s_{i,j}$  are not symmetric in the Bethe roots. To overcome this, we use a symmetrization procedure to write  $S_n$  as:

$$S_n \propto \det_{1 \leq i, j \leq n} (\tilde{s}_{i,j}), \quad (3)$$

where now  $\tilde{s}_{i,j}$  depend on the Bethe roots symmetrically. This is the same symmetrization which is used to show that the Jacobi-Trudi determinant of homogenous symmetric polynomials is equal to a Schur function [12].

Next we rewrite  $\tilde{s}_{i,j}$  in terms of the Baxter's  $Q$ -function and the  $F$ -function, that we define as follows:

$$F_n(x) = \prod_{i=1}^{2n} (x - q^2 z_i). \quad (4)$$

Next, we use the T-Q relation [2] of the twisted XXZ spin chain. It allows to write the matrix elements  $\tilde{s}_{i,j}$  back in terms of the eigenvalues  $\tau_n$  without any derivatives. Note that

$\tau_n$  is symmetric with respect to the Bethe roots. Our expression for the scalar products in terms of  $\tau_n$  (not the derivatives of  $\tau_n$ ) resemble the expression obtained in the framework of separation of variables [16] and also expressions for the XXX spin-1/2 chain [11]. Until this point we assume generic  $q$ , generic twist  $\kappa$  and also the numbers  $N$  and  $n$ .

The second important aspect is the knowledge of the ground state eigenvalue (i.e. when  $N = 2n$ ) of the transfer matrix  $\tau_n$  at  $q^3 = 1$  and  $\kappa = q^2$ . This eigenvalue is given as a ratio of two  $F$  functions which depend only on a parameter  $\mu$  and the inhomogeneities  $z$ 's. Thus we rewrite the scalar product  $S_n$  completely in terms of the  $F$ -functions. Expanding the  $F$  functions in terms of the elementary symmetric polynomials we observe that our resulting matrix is, in fact, a product of two matrices. The scalar product  $S_n$  becomes a product of two determinants one of which is the Weyl formula for a Schur function of  $\mu_1, \dots, \mu_n$  while the other one is the dual Jacobi-Trudi formula for a Schur function of  $z_1, \dots, z_{2n}$ .

The outline of the paper is the following. We give the introductory details on the Algebraic Bethe ansatz for the XXZ spin- $\frac{1}{2}$  model in the second section. Then, in the third section, we write in detail the first aspect described above. The second aspect of our derivation with the result for the scalar product are present in the fourth section. In the end of this section we include the computation of the norm of the ground state. The fifth section contains the computation of the  $\sigma_m^z$  expectation value. After that we summarize the result of our work and discuss few open problems in the conclusion section. The appendix explains how to go from symmetric polynomials of the Bethe roots to the symmetric polynomials of the rapidities.

## 2 The XXZ Heisenberg spin- $\frac{1}{2}$ inhomogeneous finite chain

In this chapter we will introduce the model and set the notations and conventions. The algebraic Bethe ansatz for this model can be found e.g. in [5]. The XXZ Heisenberg model is defined by the following Hamiltonian:

$$H_{XXZ} = J \sum_{i=1}^N \{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta (\sigma_i^z \sigma_{i+1}^z - 1) \}, \quad (5)$$

where  $\sigma_i^x$ ,  $\sigma_i^y$  and  $\sigma_i^z$  are the standard Pauli matrices and the conditions  $\sigma_{N+1}^x = \sigma_1^x$ ,  $\sigma_{N+1}^y = \sigma_1^y$  and  $\sigma_{N+1}^z = \sigma_1^z$  reflect the periodicity of the system. Now, we switch to

$$\sigma_i^+ = (\sigma_i^x + i\sigma_i^y)/2, \quad \sigma_i^- = (\sigma_i^x - i\sigma_i^y)/2, \quad (6)$$

and the conditions  $\sigma_{N+1}^+ = e^{i\phi} \sigma_1^+$ ,  $\sigma_{N+1}^- = e^{-i\phi} \sigma_1^-$  and  $\sigma_{N+1}^z = \sigma_1^z$  reflect the twisted periodicity with the twist parameter  $\phi$ .

The Algebraic Bethe ansatz is formulated using the six vertex model. This model is defined by the  $R$ -matrix

$$R(\mu, t) = \begin{pmatrix} a(\mu, t) & 0 & 0 & 0 \\ 0 & b(\mu, t) & c(\mu, t) & 0 \\ 0 & c(\mu, t) & b(\mu, t) & 0 \\ 0 & 0 & 0 & a(\mu, t) \end{pmatrix}, \quad (7)$$

with the following weights

$$\begin{aligned} a(\mu, t) &= \frac{q^2 \mu - t}{q(\mu - t)}, \quad b(\mu, t) = 1, \\ c(\mu, t) &= \frac{(q^2 - 1)\sqrt{\mu t}}{q(\mu - t)}. \end{aligned} \quad (8)$$

The  $R$ -matrix acts as a linear operator on two linear spaces  $V_i$  and  $V_j$  which are isomorphic to  $\mathbb{C}^2$ . To the spaces  $V_i$  and  $V_j$  we associate the spectral parameters  $\mu_i$  and  $\mu_j$ , hence the  $R$ -matrix acting on  $V_i \otimes V_j$  depends on  $\mu_i$  and  $\mu_j$  and is denoted by  $R_{i,j}(\mu_i, \mu_j)$ . Such an  $R$ -matrix satisfies the Yang-Baxter equation:

$$R_{i,j}(\mu_i, \mu_j)R_{i,k}(\mu_i, \mu_k)R_{j,k}(\mu_j, \mu_k) = R_{j,k}(\mu_j, \mu_k)R_{i,k}(\mu_i, \mu_k)R_{i,j}(\mu_i, \mu_j). \quad (9)$$

The two dimensional Hilbert space  $\mathcal{H}_j$  of  $SU(2)$  spin-1/2 chain corresponds to the  $i$ 'th site of the chain. We construct the  $L$  matrix acting on the site  $j$ :

$$L_j(\mu, z_j) = R_{0,j}(\mu, z_j). \quad (10)$$

Thus the  $L$  operator acts on the tensor product  $\mathbb{C}^2 \otimes \mathcal{H}_j$  and the parameter  $z_j$  is an arbitrary complex number called the inhomogeneity (spectral parameter) associated to the space  $\mathcal{H}_j$ . Now we use the  $L$  operators to construct the monodromy matrix:

$$T(\mu) = J L_N(\mu, z_N) \dots L_1(\mu, z_1). \quad (11)$$

The matrix  $J$  is the diagonal matrix that includes the twist. The matrix (11) is represented in the space with the label 0, called the auxiliary space, as a two dimensional matrix:

$$T(\mu) = \begin{pmatrix} A(\mu) & B(\mu) \\ \kappa C(\mu) & \kappa D(\mu) \end{pmatrix}, \quad (12)$$

where the matrix elements  $A$ ,  $B$ ,  $C$  and  $D$  are linear operators acting on the Hilbert space  $\mathcal{H} = \otimes_{j=1}^N \mathcal{H}_j$  of the chain of length  $N$ . The parameter  $\kappa$  is the twist parameter which is related to the parameter  $\phi$ . The operators  $A$ ,  $B$ ,  $C$  and  $D$  form the Yang-Baxter algebra whose commutation relations follow from the RTT relation:

$$R_{1,2}(\mu_1, \mu_2)T_1(\mu_1)T_2(\mu_2) = T_2(\mu_2)T_1(\mu_1)R_{1,2}(\mu_1, \mu_2), \quad (13)$$

where we used the notation  $T_1(\mu) = T(\mu) \otimes Id$  and  $T_2(\mu) = Id \otimes T(\mu)$ .

The trace of the monodromy matrix eq.(12) over the auxiliary space defines the transfer matrix  $\mathcal{T}(\mu) = A(\mu) + \kappa D(\mu)$ . Thanks to the Yang-Baxter equation the transfer matrices at different values of the spectral parameter  $\mu$  commute. The Hamiltonian  $H_{XXZ}$  can be written in terms of the transfer matrix  $\mathcal{T}(\mu)$  hence the problem turns into the diagonalization of  $\mathcal{T}(\mu)$  for all values of  $\mu$ .

In the algebraic Bethe ansatz the eigenvectors of the transfer matrix can be written in terms of the  $B$ -operators acting on the reference state  $|0\rangle$ . This state must have the following properties:

$$\begin{aligned} A(\mu)|0\rangle &= a(\mu)|0\rangle, \\ D(\mu)|0\rangle &= d(\mu)|0\rangle, \\ C(\mu)|0\rangle &= 0, \\ B(\mu)|0\rangle &\neq 0. \end{aligned} \quad (14)$$

The functions  $a(\mu)$  and  $d(\mu)$  are defined as:

$$a(\mu) = \prod_{i=1}^N a(\mu, z_i), \quad d(\mu) = \prod_{i=1}^N b(\mu, z_i). \quad (15)$$

The  $|0\rangle$  state is totally ferromagnetic in the case of the XXZ model. The eigenvectors of the transfer matrix are obtained by the successive action of the  $B$ -operators on the reference state:

$$\psi_n = \prod_{i=1}^n B(\zeta_i | z_1, \dots, z_N) |0\rangle, \quad (16)$$

where the parameters  $\zeta_1, \dots, \zeta_n$  satisfy the Bethe equations:

$$\prod_{i=1}^N a(\zeta_k, z_i) \prod_{\substack{i=1 \\ i \neq k}}^n \frac{a(\zeta_i, \zeta_k)}{c(\zeta_i, \zeta_k)} - (-1)^n \kappa \prod_{i=1}^N b(\zeta_k, z_i) \prod_{\substack{i=1 \\ i \neq k}}^n \frac{a(\zeta_k, \zeta_i)}{c(\zeta_k, \zeta_i)} = 0, \quad k = 1, 2, \dots, n. \quad (17)$$

Taking into account (8) we may write these equations simply:

$$\prod_{i=1}^N a(\zeta_k, z_i) \prod_{\substack{i=1 \\ i \neq k}}^n a(\zeta_i, \zeta_k) - \kappa \prod_{i=1}^N b(\zeta_k, z_i) \prod_{\substack{i=1 \\ i \neq k}}^n a(\zeta_k, \zeta_i) = 0, \quad k = 1, 2, \dots, n. \quad (18)$$

A state (16) with the parameters  $\zeta_1, \dots, \zeta_n$  that satisfy (17) is called the  $n$ -particle Bethe state and the parameters  $\zeta_1, \dots, \zeta_n$  are called the Bethe roots. We will reserve the indexed letter  $\zeta$  in what follows to denote the Bethe roots, the parameters  $z_1, \dots, z_N$  will denote the inhomogeneities of the system and the parameters  $\mu_1, \dots, \mu_n$  will be another set of free parameters which is necessary in order to write the scalar products. The transfer matrix eigenvalue corresponding to the  $n$ -particle state is

$$\tau_n(\mu) = \prod_{i=1}^N a(\mu, z_i) \prod_{i=1}^n a(\zeta_i, \mu) + \kappa \prod_{i=1}^N b(\mu, z_i) \prod_{i=1}^n a(\mu, \zeta_i). \quad (19)$$

The scalar products of states are defined as

$$S_n(\mu_1, \dots, \mu_n; \zeta_1, \dots, \zeta_n) = \langle 0 | \prod_{i=1}^n C(\mu_i) \prod_{i=1}^n B(\zeta_i) | 0 \rangle. \quad (20)$$

Here, as mentioned before  $\zeta_1, \dots, \zeta_n$  are the Bethe roots and  $\mu_1, \dots, \mu_n$  are free parameters. If the parameters  $\mu$  also satisfy the Bethe equations, then the product of the  $C$  operators acting on the dual reference state

$$\langle 0 | \prod_{i=1}^n C(\mu_i) \quad (21)$$

is the dual Bethe state. If we want to compute the expectation value of an operator  $\mathcal{O}$ :

$$\langle \mathcal{O} \rangle = \langle 0 | \prod_{i=1}^{n_0} C(\mu_i) \mathcal{O} \prod_{i=1}^n B(\zeta_i) | 0 \rangle, \quad (22)$$

and, say, we computed the action of  $\mathcal{O}$  on the dual Bethe state written as a combination of dual states

$$\langle 0 | \prod_{i=1}^{n_0} C(\mu_i) \mathcal{O} = \sum_k \theta_k \langle 0 | \prod_{i=1}^{n_k} C(\nu_i^{(k)}), \quad (23)$$

where  $\nu_i^{(k)}$  are some numbers, then the computation of  $\langle \mathcal{O} \rangle$  boils down to the computation of the scalar product (20).

Fortunately, the scalar products (20) have a nice determinantal representation [18, 19]. Let us introduce the matrix elements  $\Omega_{j,k}$  which depend on the system length  $N$  and the number of particles  $n$ :

$$\Omega_{j,k} = \frac{\partial \tau_n(\mu_k | \zeta_1, \dots, \zeta_n)}{\partial \zeta_j} \prod_{i=1}^n c^{-1}(\zeta_i, \mu_k). \quad (24)$$

Then the scalar product  $S_n$  is

$$S_n(\mu_1, \dots, \mu_n; \zeta_1, \dots, \zeta_n) = \frac{(q^2 - 1)^n}{2^n} \prod_{i < j} c(\mu_j, \mu_i) c(\zeta_i, \zeta_j) \det_{1 \leq j, k \leq n} \Omega_{j,k}. \quad (25)$$

Let us rewrite slightly this expression. We introduce two functions that we will use later on:

$$F_N(x) = \prod_{i=1}^N (x - q^2 z_i), \quad (26)$$

and

$$Q_n(x) = \prod_{i=1}^n (x - \zeta_i). \quad (27)$$

Where  $Q_n(x)$  is nothing but the Baxter's  $Q$ -function [2] corresponding to the  $n$ -particle state. The roots of the  $Q$ -polynomial are the Bethe roots. We will often omit the indices in  $F$  and  $Q$ , since usually this is clear from the context. In terms of  $F$  and  $Q$  the matrix elements  $\Omega_{j,k}$  become:

$$\Omega_{j,k} = \frac{-2(q^2 - 1)\zeta_j}{(1 - q^2)^n F(q^2 \mu_k) \mu_k^{n/2-1} \prod_{i \neq j} \zeta_i^{1/2}} \times c_{j,k}, \quad (28)$$

where we gathered the nontrivial part of the matrix elements  $\Omega_{j,k}$  into  $c_{j,k}$ :

$$c_{j,k} = \frac{1}{\mu_k - \zeta_j} \left( \frac{Q(q^{-2} \mu_k) F(q^4 \mu_k)}{\mu_k - q^2 \zeta_j} + \kappa \frac{Q(q^2 \mu_k) F(q^2 \mu_k)}{\zeta_j - q^2 \mu_k} \right). \quad (29)$$

The prefactor of  $c_{j,k}$  in (28) can be taken out from the sign of the determinant in (25), so now we need to compute:

$$\tilde{S}_n = \frac{1}{\prod_{i < j} (\zeta_j - \zeta_i)(\mu_j - \mu_i)} \det_{1 \leq j, k \leq n} c_{j,k}. \quad (30)$$

### 3 Symmetric expression for the scalar products

In this section we show the derivation of a symmetric expression for the matrix  $c_{j,k}$ . We first split the matrix  $c_{j,k}$  into two parts proportional to  $a_{j,k}$  and  $b_{j,k}$ :

$$a_{j,k} = \frac{1}{(\mu_k - \zeta_j)(\mu_k - q^2 \zeta_j)}, \quad (31)$$

$$b_{j,k} = \frac{1}{(\mu_k - \zeta_j)(\zeta_j - q^2 \mu_k)}, \quad (32)$$

so that

$$c_{j,k} = Q(q^{-2} \mu_k) F(q^4 \mu_k) a_{j,k} + \kappa Q(q^2 \mu_k) F(q^2 \mu_k) b_{j,k}. \quad (33)$$

Now we can symmetrize separately the  $a_{j,k}$  and the  $b_{j,k}$  parts. In order to do that we follow the procedure that allows to show, for example, that a Schur function can be written as a determinant of the homogenous symmetric functions [12]. First, define the operators  $\mathcal{R}_i$  which act on a  $n \times n$  matrix  $A$  as follows:

$$(\mathcal{R}_k A)_{i,j} = A_{i,j} - \delta_{i,k} A_{i+1,j}. \quad (34)$$

This operator subtracts two rows  $i$  and  $i+1$  which have the same form except that one depends on  $\zeta_i$  and the other one depends on  $\zeta_{i+1}$ . The result of such subtraction is proportional to  $\zeta_i - \zeta_{i+1}$  with the proportionality factor being symmetric in the interchange of  $\zeta_i$  and  $\zeta_{i+1}$ . Let  $A$  have components either  $a_{i,j}$  or  $b_{i,j}$ , then we apply to it the product of the  $\mathcal{R}_i$  operators:

$$\begin{aligned} \mathcal{R}_{n-1} \dots \mathcal{R}_1 A &= A^{(1)} \prod_{i=1}^{n-1} (\zeta_i - \zeta_{i+1}), \\ \mathcal{R}_{n-2} \dots \mathcal{R}_1 A^{(1)} &= A^{(2)} \prod_{i=1}^{n-2} (\zeta_i - \zeta_{i+2}), \\ &\vdots \\ \mathcal{R}_1 A^{(n-1)} &= A^{(n)} (\zeta_1 - \zeta_n). \end{aligned} \quad (35)$$

The components of the resulting matrix  $A^{(n)}$ , apart from the first row, are not yet symmetric in the Bethe roots. On the way to  $A^{(n)}$  we lost exactly the Vandermonde determinant of the Bethe roots as we wanted. The matrix elements of  $A^{(n)}$  (recall that those are either  $a_{j,k}$  or  $b_{j,k}$ ) significantly differ in their form from one row to another since we made a different number of operations on different rows. We need to apply another transformation to the matrix  $A^{(n)}$ . Define the operators  $\mathcal{R}_i(x)$  which act on a  $n \times n$  matrix in the following way

$$(\mathcal{R}_k(x) A)_{i,j} = A_{i,j} + \delta_{i,k} x A_{i-1,j}. \quad (36)$$

And now we do something similar as before

$$\begin{aligned} \mathcal{R}_2(\zeta_1) \dots \mathcal{R}_n(\zeta_{n-1}) A^{(n)} &= A^{(n,1)}, \\ \mathcal{R}_3(\zeta_1) \dots \mathcal{R}_n(\zeta_{n-2}) A^{(n,1)} &= A^{(n,2)}, \\ &\vdots \\ \mathcal{R}_n(\zeta_1) A^{(n,n-1)} &= A^{(n,n)}. \end{aligned} \quad (37)$$

The matrix  $A^{(n,n)}$  is symmetric in the Bethe roots as we wanted. To see how do the matrix elements of  $a$  and  $b$  look we need to express the transformations of the first step (35) and the transformations of the second step (37) in terms of matrices. An operator  $\mathcal{R}_i$  can be viewed as a simple matrix, hence the transformation, for example, of the first line of (35) will be a product of such matrices divided by the prefactor of the  $A^{(1)}$  matrix. The transformation of the first step (35) will be the product of the matrices corresponding to the transformations of each line in (35). Similar logic applies to the second step (37). We denote these matrices by  $\rho_1$  and  $\rho_2$  respectively:

$$\rho_1 = \begin{pmatrix} \frac{1}{(\zeta_1 - \zeta_2)(\zeta_1 - \zeta_3) \dots (\zeta_1 - \zeta_n)} & \frac{1}{(\zeta_2 - \zeta_1)(\zeta_2 - \zeta_3) \dots (\zeta_2 - \zeta_n)} & \cdots & \frac{1}{(\zeta_n - \zeta_1)(\zeta_n - \zeta_2) \dots (\zeta_n - \zeta_{n-1})} \\ 0 & \frac{1}{(\zeta_2 - \zeta_3) \dots (\zeta_2 - \zeta_n)} & \cdots & \frac{1}{(\zeta_n - \zeta_2) \dots (\zeta_n - \zeta_{n-1})} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\zeta_n - \zeta_{n-1}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (38)$$

$$\rho_2 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ h_1(\zeta_1) & 1 & 0 & \dots & 0 \\ h_2(\zeta_1) & h_1(\zeta_1, \zeta_2) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ h_{n-1}(\zeta_1) & h_{n-2}(\zeta_1, \zeta_2) & h_{n-3}(\zeta_1, \zeta_2, \zeta_3) & \dots & 1 \end{pmatrix}, \quad (39)$$

where  $h_k$  are the homogeneous symmetric functions:

$$h_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}, \quad (40)$$

which have, in particular, the following useful property:

$$h_{i-1}(x_1, \dots, x_m) = \frac{h_i(x_1, \dots, \hat{x}_l, \dots, x_m) - h_i(x_1, \dots, \hat{x}_k, \dots, x_m)}{x_k - x_l}, \quad (41)$$

where  $\hat{x}$  means the absence of the corresponding variable. Multiplying the two transformations  $\rho_1$  and  $\rho_2$  and using the property (41) we obtain the full transformation  $\rho = \rho_2 \rho_1$ :

$$\rho_{i,j} = \frac{\zeta_j^{i-1}}{\prod_{k \neq j} (\zeta_j - \zeta_k)}. \quad (42)$$

This is a linear transformation, so we act separately on  $a_{i,j}$  and  $b_{i,j}$  and then add up the results with the appropriate coefficients according to (33). Let us take first the matrix  $a_{i,j}$  and multiply it by the matrix  $\rho$  from the left. We obtain:

$$\rho_{i,j} a_{j,k} = \sum_{j=1}^n \frac{\zeta_j^{i-1}}{\prod_{l \neq j} (\zeta_j - \zeta_l)} \times \frac{1}{(\mu_k - \zeta_j)(\mu_k - q^2 \zeta_j)}. \quad (43)$$

Let us multiply (43) by the product  $\mu_k^{2-i} Q(\mu_k) Q(q^{-2} \mu_k)$ , we get

$$\sum_{j=1}^n \frac{\zeta_j^{i-1}}{\prod_{l \neq j} (\zeta_j - \zeta_l)} \times \frac{\mu_k^{2-i} Q(\mu_k) Q(q^{-2} \mu_k)}{(\mu_k - \zeta_j)(\mu_k - q^2 \zeta_j)} = \sum_{j=1}^n q^{-2} \zeta_j^{i-1} \mu_k^{2-i} \prod_{l \neq j} \frac{(\mu_k - \zeta_l)(\mu_k q^{-2} - \zeta_l)}{(\zeta_j - \zeta_l)}. \quad (44)$$



The right hand side here is the Lagrange polynomial of a function, let us call it  $f(\mu_k)$ . Now we set  $\mu_k = \zeta_i$  for  $i = 1, \dots, n$ . From (44) we obtain

$$f(\zeta_i) = \frac{1}{1 - q^2} Q(q^{-2}\zeta_1). \quad (45)$$

If we set  $\mu_k = q^2\zeta_i$  for  $i = 1, \dots, n$ , then

$$f(q^2\zeta_i) = q^{2-2i} \frac{1}{q^2 - 1} Q(q^2\zeta_1). \quad (46)$$

It is easy to check that

$$f(\mu) = \frac{Q(\mu)q^{2-2i} - Q(q^{-2}\mu)}{q^2 - 1}, \quad (47)$$

satisfies the above  $2n$  recurrence relations (45) and (46) being a polynomial of degree  $n$ , hence the Lagrange polynomial in (44) is equal to the function (47). The matrix elements  $a_{i,j}$  after the transformation take the form:

$$\rho_{i,j} a_{j,k} = \mu_k^{i-2} \frac{Q(\mu_k)q^{2-2i} - Q(q^{-2}\mu_k)}{(q^2 - 1)Q(\mu_k)Q(q^{-2}\mu_k)}. \quad (48)$$

Now we do the same with the  $b_{i,j}$  matrix. The result will be

$$\rho_{i,j} b_{j,k} = \mu_k^{i-2} \frac{Q(\mu_k)q^{2i-2} - Q(q^2\mu_k)}{(q^2 - 1)Q(\mu_k)Q(q^2\mu_k)}. \quad (49)$$

Finally, we can write the action of the transformation  $\rho$  on the matrix elements  $c_{i,j}$ . We will call  $\tilde{c}_{i,j}$  the matrix elements of the transformed Slavnov matrix  $\tilde{c}_{i,k} = \rho_{i,j} c_{j,k}$

$$\tilde{c}_{i,k} = \frac{\mu_k^{i-2}}{(q^2 - 1)} \left( \frac{F(q^4\mu_k)}{Q(\mu_k)} (q^{2-2i} Q(\mu_k) - Q(q^{-2}\mu_k)) - \kappa \frac{F(q^2\mu_k)}{Q(\mu_k)} (Q(q^2\mu_k) - q^{2i-2} Q(\mu_k)) \right). \quad (50)$$

This expression already satisfies our needs. It is symmetric in the Bethe roots, and even more than that. We do not need to express it as elementary symmetric polynomials of the Bethe roots and then use the  $Q$  function, since it is already written in terms of the  $Q$  function. The determinant

$$\tilde{S}_n = \prod_{i < j} (\mu_i - \mu_j)^{-1} \det_{1 \leq i, j \leq n} \tilde{c}_{i,j} \quad (51)$$

is written in a nice form, since it depends only on the  $Q$  operators and a simple factorized function  $F$ . Similar formulae for the scalar product appear in the separation of variables approach, see [16].

Now let us make (50) even nicer. For that we need to use the  $T - Q$  relation for the six vertex model [2]. In this relation the transfer matrix eigenvalues and the  $Q$  operator satisfy a quadratic relation, in our notations it reads

$$\tau(\mu)Q(\mu) = q^{-n} \frac{F(q^4\mu)}{F(q^2\mu)} Q(q^{-2}\mu) + q^{-n} \kappa Q(q^2\mu). \quad (52)$$

We can use this expression to simplify the matrix elements  $\tilde{c}_{i,j}$ . After this simplifications the  $Q$ -functions disappear. Indeed, the sum of the two terms in (50) which contain  $Q(q^{-2}\mu)$  and  $Q(q^2\mu)$  is proportional to  $\tau(\mu)Q(\mu)$ , the denominator of  $Q(\mu)$  cancels in every term in this expression, therefore the remaining depends on the transfer matrix eigenvalue  $\tau(\mu)$  and the polynomials  $F$ :

$$\tilde{c}_{i,k} = \frac{\mu_k^{i-2}}{(q^2 - 1)} \left( q^{2-2i} F(q^4 \mu_k) + F(q^2 \mu_k) (q^{-2+2i} \kappa - q^n \tau(\mu_k)) \right), \quad (53)$$

and recalling that the prefactor in (28) contains  $1/F(q^2 \mu_k)$ , we write:

$$\bar{c}_{i,k} = \frac{\mu_k^{i-2}}{(q^2 - 1)} \left( q^{2-2i} \frac{F(q^4 \mu_k)}{F(q^2 \mu_k)} + q^{-2+2i} \kappa - q^n \tau(\mu_k) \right), \quad (54)$$

Finally, the Slavnov product reads:

$$\tilde{S}_n = \frac{1}{\prod_{i < j} (\mu_j - \mu_i)} \det_{1 \leq i, k \leq n} \mu_k^{i-2} \left( q^{2-2i} \frac{F(q^4 \mu_k)}{F(q^2 \mu_k)} + q^{-2+2i} \kappa - q^n \tau(\mu_k) \right). \quad (55)$$

This expression involves the eigenvalue of the transfer matrix which contains all the dependence on the Bethe roots and is symmetric in their interchange. When the first set of variables in the scalar product are the Bethe roots ( $\mu_1, \dots, \mu_n$  in our notation) a similar formula to the (55) can be obtained.

## 4 Scalar product at $q^3 = 1$

In this section we will set  $q^3 = 1$ ,  $\kappa = q^2$  and consider the scalar product in which  $N = 2n$ , i.e. the one for the ground state vector. In this special case we know the eigenvalue of the transfer matrix. We will substitute it in the matrix elements  $\tilde{c}_{i,j}$  in the form (53) and after some simplifications obtain a product of two Schur functions in which one Schur function depends on the inhomogeneities while the other one on the free parameters  $\mu$ .

The ground state eigenvalue of the transfer matrix reads [7]:

$$\tau(\mu) = -q^{2n+1} \frac{F(\mu)}{F(q^2 \mu)}. \quad (56)$$

Matrix elements  $\tilde{c}_{i,k}$  become

$$\tilde{c}_{j,k} = \frac{\mu_k^{j-2}}{q^2 - 1} (q^{2-2j} F(q^4 \mu_k) + q^{2j} F(q^2 \mu_k) + q F(\mu_k)). \quad (57)$$

The functions  $F$  are the generating functions of the elementary symmetric polynomials:

$$F_n(x) = \sum_{i=0}^{2n} (-q^2)^{2n-i} x^i e_{2n-i}(z_1, \dots, z_{2n}), \quad (58)$$

$$e_k(z_1, \dots, z_m) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq m} z_{i_1} z_{i_2} \dots z_{i_k}, \quad (59)$$

$$e_k(z_1, \dots, z_m) = 0, \quad \text{for } k < 0 \text{ or } k > m. \quad (60)$$

Substituting (58) in (57) we obtain:

$$\tilde{c}_{j,k} = \frac{\mu_k^{j-2} q^{4n}}{q^2 - 1} \sum_{s=0}^{2n} (-1)^s \mu_k^s e_{2n-s} (q^{1-2s} + q^{2j} + q^{2+2s+2j}),$$

which can be rewritten as

$$\tilde{c}_{j,k} = \frac{\mu_k^{j-2} q^{n-j}}{q^2 - 1} \sum_{s=0}^{2n} (-1)^s \mu_k^s e_{2n-s} (q^{1+s+j} + q^{-(1+s+j)} + 1), \quad (61)$$

where we used  $q^2 = q^{-1}$  and also assumed  $e_i = e_i(z_1, \dots, z_{2n})$ . The factor  $q^{1+s+j} + q^{-(1+s+j)} + 1$  in the last expression is equal to zero unless  $j + s + 1 = 3m$ , with  $m \in \mathbb{Z}$ , in which case it is equal to 3. Noticing this, we change the summation  $s = 3m - j - 1$ :

$$\tilde{c}_{j,k} = \frac{3q^{n-j}}{q^2 - 1} \sum_{m=\lceil \frac{j+1}{3} \rceil}^{\lfloor \frac{2n+j+1}{3} \rfloor} (-1)^{j+m+1} \mu_k^{3m-3} e_{2n-3m+j+1}. \quad (62)$$

Because of the property (60) we can put the initial value of the summation in  $m$  to 1. For the first and second row ( $j = 1, 2$ ) this is already true, but for the remaining rows the summation will start at a value  $m > 1$  and due to (60) we can extend it to  $m = 1$ . When the value of  $m$  becomes higher than  $n$  all terms vanish by the same property (60), so we can simply put the upper summation limit to be  $n$ . Hence we can write:

$$\tilde{c}_{j,k} = \frac{3q^{n-j}}{q^2 - 1} \sum_{m=1}^n (-1)^{j+m+1} \mu_k^{3m-3} e_{2n-3m+j+1}. \quad (63)$$

The last expression is nothing but the product of two  $n \times n$  matrices

$$A_{k,m} = \mu_k^{3m-3}, \quad B_{m,j} = e_{2n-3m+j+1}, \quad (64)$$

multiplied by a prefactor that we can take out of the determinant. The matrix  $A$  is the Schur polynomial of the partition  $Y_n = \{2n - 2i\}_{i=1}^n$  by the Weyl formula for the  $GL(N)$  characters:

$$\det A = s_{Y_n}(\mu_1, \dots, \mu_n). \quad (65)$$

This Schur function has a simple factorized form:

$$s_{Y_n}(\mu_1, \dots, \mu_n) = \prod_{1 \leq i < j \leq n} (\mu_i^2 + \mu_i \mu_j + \mu_j^2). \quad (66)$$

The matrix  $B$  is the Schur polynomial of the partition  $\tilde{Y}_n = \{n, n-1, n-1, \dots, 1, 1, 0\}$  by the dual Jacobi-Trudi identity

$$\det B = s_{\tilde{Y}_n}(z_1, \dots, z_{2n}). \quad (67)$$

The determinant  $S_n$  becomes:

$$S_n = \frac{3^n q^n \prod_{i=1}^n \mu_i^{1/2} \zeta_i^{1/2}}{\prod_{i=1}^n q^{-4n} F(q^2 \mu_i)} s_{Y_n}(\mu_1, \dots, \mu_n) s_{\tilde{Y}_n}(z_1, \dots, z_{2n}). \quad (68)$$

Because of the simplicity of this expression we hope that it will be helpful in the problem of the computation of correlation functions.

Before turning to the problem of the computation of the form factor  $\langle \sigma^z \rangle$  we derive the normalization of the ground state  $\mathcal{N}_n$ . It was computed previously in [3] and also in [20]. This quantity is obtained from the eq.(68) by assuming  $\mu_1, \dots, \mu_n$  to be the Bethe roots. In the appendix we will show that a Schur function  $s_\pi(\zeta_1, \dots, \zeta_n)$  of a partition  $\pi$ , which is a symmetric polynomial of the Bethe roots  $\zeta_i$ , is equal to some polynomial  $p_\pi(z_1, \dots, z_{2n})$  which is labeled by the same partition  $\pi$ , but depends on the inhomogeneities. It turns out, however, that in the particular case of the Schur function of the partition  $Y_n$ , which appears in (68), we can write  $s_{Y_n}$  in a more explicit form. Setting  $\mu_i = \zeta_i$  in (68) we get:

$$\mathcal{N}_n = \frac{3^n q^n \prod_{i=1}^n \zeta_i}{\prod_{i=1}^{2n} Q(z_i)} s_{Y_n}(\zeta_1, \dots, \zeta_n) s_{\tilde{Y}_n}(z_1, \dots, z_{2n}). \quad (69)$$

Now we need to know what is  $s_{Y_n}(\zeta_1, \dots, \zeta_n) \prod_{i=1}^n \zeta_i$ . Taking into account (66) and that  $q^3 = 1$  we can rewrite this as:

$$s_{Y_n}(\zeta_1, \dots, \zeta_n) \prod_{i=1}^n \zeta_i = (q-1)^{-n} \prod_{i=1}^n Q(q\zeta_i). \quad (70)$$

Now we use the identity

$$\prod_{i=1}^n Q(q\zeta_i) = q^{2n^2-n} (1-q)^n \frac{s_{\tilde{Y}_n}(z_1, \dots, z_{2n}) s_{Y'_n}^2(z_1, \dots, z_{2n})}{s_{Y_n^0}(z_1, \dots, z_{2n})} \prod_{i=1}^{2n} \frac{Q(z_i)}{F(z_i)}, \quad (71)$$

where the partition  $Y_0$  is simply  $Y_0 = \{1, 1, \dots, 1\}$  and the partition  $Y'_{2n}$  is another staircase partition:  $Y'_{2n} = \{n, n, \dots, 1, 1\}$ . The entries of the equation (71) satisfy certain recurrence relations upon setting  $z_i = qz_j$ . The explicit form of  $Q$  in terms of inhomogeneities is known [7], and we remind it in the appendix (100). These two ingredients are enough to prove (71). It would be, however, much more usefull to have a direct proof of the eq.(71).

The product of the polynomials  $F$  in the denominator in (71) can be expressed through the the product of Schur function of partitions  $Y_{2n}$  and of  $Y_n^0$ , hence, omitting the irrelevant prefactor depending on  $q$ , we get:

$$\mathcal{N}_n = \frac{s_{\tilde{Y}_n}^2(z_1, \dots, z_{2n}) s_{Y'_{2n}}^2(z_1, \dots, z_{2n})}{s_{Y_n^0}^2(z_1, \dots, z_{2n}) s_{Y_{2n}}(z_1, \dots, z_{2n})}. \quad (72)$$

## 5 Expectation value of $\sigma_m^z$

Frist, we need to renormalize our  $R$ -matrix (7) in order to avoid singularities during the computations. We divide all the weights  $a$ ,  $b$  and  $c$  by the weight  $a$  and by abuse of notation denote the new weights by the same letters  $a$ ,  $b$  and  $c$ .

The operator  $\sigma_m^z$  can be written in the F-basis [13, 9]:

$$\sigma_m^z = \prod_{i < m} \mathcal{T}(z_i) (A(z_m) - \kappa D(z_m)) \prod_{i > m} \mathcal{T}(z_i), \quad (73)$$

Now we sandwich this expression with the left and the right Bethe states. The result we can write as:

$$\langle \sigma_m^z \rangle = 2 \prod_{i=1}^{m-1} \tau_n(z_i) \prod_{i=m+1}^n \tau_n(z_i) \langle 0 | \prod_{i=1}^n C(\mu_i) A(z_m) \prod_{i=1}^n B(\zeta_i) | 0 \rangle - \kappa S_n(\{\mu\}; \{\zeta\}). \quad (74)$$

Using the Yang-Baxter algebra given by the RTT relations (13) we can commute  $A(z_m)$  in the first term through the  $B$ -operators. Noticing also that the prefactor of the first term in (74) is equal to  $2\kappa/\tau_n(z_m)$  due to the property coming from the Bethe equations:

$$\prod_{i=1}^N \tau_n(z_i) = \kappa, \quad (75)$$

we can write the expectation value  $\sigma_m^z$  in the form:

$$\langle \sigma_m^z \rangle = S_n(\{\mu\}; \{\zeta\}) - \sum_{a=1}^n f_a S_n(\{\mu\}; \zeta_1, \dots, \hat{\zeta}_a, \dots, \zeta_n, z_m), \quad (76)$$

where  $f_a$  are some coefficients depending on the Bethe roots (here, both  $\mu_j$  and  $\zeta_j$  are the Bethe roots). We omitted the overall factor of  $\kappa$  in the last equation. More explicitly, the equation (76) reads:

$$\langle \sigma_m^z \rangle = S_n(\zeta_1, \dots, \zeta_n) - 2 \prod_{i=1}^n \frac{b(\zeta_i, z_m)}{a(\zeta_i, z_m)} \sum_{i=1}^n \frac{c(\zeta_i, z_m)}{b(\zeta_i, z_m)} \prod_{j \neq i} \frac{a(\zeta_j, \zeta_i)}{b(\zeta_j, \zeta_i)} S_n(z_m, \zeta_1, \dots, \hat{\zeta}_i, \dots, \zeta_n). \quad (77)$$

Substituting the weights  $a$ ,  $b$  and  $c$ , the scalar product as in (55) and performing some algebraic manipulations, we obtain:

$$\langle \sigma_m^z \rangle = \tilde{S}_n \left( 1 + 6q \frac{Q(z_m)Q(q^2 z_m)}{F(qz_m)} G_n \right), \quad (78)$$

$$G_n = \sum_{i=1}^n \frac{\zeta_i z_m F(q\zeta_i)}{(\zeta_i - z_m)(q\zeta_i - z_m)(\zeta_i - qz_m)Q(q^2 \zeta_i) \prod_{j \neq i} (\zeta_i - \zeta_j)}. \quad (79)$$

The difference between (77) and (78) is that one of the two Vandermonde determinants in the denominator of  $S_n$  is cancelled in (78), thus we write  $\tilde{S}_n$  in (78). The rational function (79) is symmetric in the Bethe roots. We don't know how to write it compactly in terms of the symmetric functions of the Bethe roots.

Both, (77) and (78) can be rewritten back in the determinant form of a single matrix. Indeed, since the matrix elements of the Slavnov matrix have the form:

$$c_{i,k} = c_i(\mu_k), \quad (80)$$

what is written, e.g. in (77) is the determinant of

$$c_i(\mu_k) + \alpha_k c_i(z_m), \quad (81)$$

where  $\alpha_k$  are the coefficients in the summation in (77).

Now let us set  $q^3 = 1$ . Using the expression for the scalar product at  $q^3 = 1$  (68) we arrive at:

$$\langle \sigma_m^z \rangle = \text{const} \frac{1}{\prod_{i < j} (\zeta_j - \zeta_i)} \det_{1 \leq j, k \leq n} \left( \frac{\zeta_k^{3j-2}}{F(q\zeta_k)} - 2 \frac{z_m^{3j-2} Q(q\zeta_k)}{F(qz_m) Q(qz_m)} \right). \quad (82)$$

where const depends on  $q$  and thus can be omitted for simplicity.

Now we can turn to the final point of our work. We must symmetrize (82) with respect to the Bethe roots  $\zeta_i$  using one or another symmetrization procedure. It is not an easy task and, we believe, there is a better way to do it than the one we choose here. Let us first write the eq.(82) as follows:

$$\langle \sigma_m^z \rangle \propto \frac{1}{\prod_{i < j} (\zeta_j - \zeta_i)} \det_{1 \leq j, k \leq n} \left( \zeta_k^{3j-2} F(qz_m) Q(qz_m) - 2 z_m^{3j-2} F(q\zeta_k) Q(q\zeta_k) \right). \quad (83)$$

This is a nicer, more symmetric, form of the determinant that we need to compute. It is also a determinant of a matrix with polynomial in  $\zeta_k$  entries.

Let us take again the symmetrization transform  $\rho$  and apply it to a vector  $(\zeta_1^l, \dots, \zeta_n^l)$ . Using the properties (41) of the complete homogenous symmetric function we obtain:

$$\sum_{j=1}^n \rho_{i,j} \zeta_j^l = h_{i+l-n}. \quad (84)$$

Applying  $\rho$  to the first term in (83) we get:

$$\sum_{k=1}^n \rho_{i,k} \zeta_k^{3j-2} = h_{i+3j-2-n}. \quad (85)$$

The second term we expand in  $\zeta_k$ :

$$F(q\zeta_k) Q(q\zeta_k) = \sum_{j=0}^{3n} (-q)^j \zeta_k^{3n-j} \gamma_j, \quad (86)$$

where  $\gamma_j$  are coefficients which depend only on the inhomogeneities and can be written as determinants. Their explicit form is given in the the appendix (105). Applying  $\rho$  to this term we get:

$$\sum_{k=1}^n \rho_{i,k} F(q\zeta_k) Q(q\zeta_k) = \sum_{j=0}^{3n} (-q)^j h_{i+2n-j} \gamma_j. \quad (87)$$

Therefore, applying the transformation  $\rho$  to the matrix elements in (83) gives us the final formula

$$\langle \sigma_m^z \rangle \propto \det_{1 \leq j, k \leq n} \left( h_{j+3k-2-n} F(qz_m) Q(qz_m) - 2 z_m^{3k-2} \sum_{i=0}^{3n} (-q)^i h_{2n+j-i} \gamma_i \right). \quad (88)$$

This formula still depends on the Bethe roots. However, each matrix element is a combination of homogenous symmetric polynomials of the Bethe roots. These and other symmetric

functions of the Bethe roots, in fact, can be written as symmetric functions in the inhomogeneities. In the appendix we explain how to do that. In particular,

$$h_i(\zeta_1, \dots, \zeta_n) = f_i(z_1, \dots, z_{2n}), \quad (89)$$

where  $f_i$  is given in (108). We believe that the expression (88) can be written in a much more simpler form. However, at this stage we don't know how to simplify it. This requires probably a different symmetrization of the eq.(83).

## 6 Discussions

We considered a particular interaction point  $\Delta = -1/2$  of the ground state of the Heisenberg XXZ spin-1/2 chain with a twist  $\kappa = q^2$ . Our aim was to approach the correlation functions for the ground state of the model starting from the notion of the scalar products. The ground state scalar product turns out to be a simple Schur function (68) which is a fully factorized polynomial (66). Using this result, in principle, we can compute the form factors and simple correlation functions corresponding to the ground state. In the last chapter we made an attempt to compute the expectation value of the  $\sigma^z$  operators. Although, we arrived at a closed form expression (88) it still does not look as nice as we would like. Despite the simplicity of the scalar product it is still no easy to obtain a good formula for the simplest correlation functions. We believe that the expression (88) can be further simplified. One, perhaps, needs to use a different symmetrization method to deal with the determinant (83).

Note that, our computations are restricted to the systems of even length. We expect that at odd lengths the same computation can be done. It is also an important problem to compute the correlation functions for the ground state of odd systems. Also, one could consider expectation values of operators with spin= $\pm 1/2$ , e.g.  $\sigma^+$  or  $\sigma^-$ . These expectation values are expected to be much easier than  $\langle \sigma^z \rangle$ . An important result of our work is the integration of the Slavnov scalar products (55), which gives a new formula (valid for generic values of  $q$ ). We started from the Slavnov representation of the scalar product, which is given by a matrix of derivatives of a fixed eigenvalue of the transfer matrix. Then applied a transformation which turned each entry of the matrix into a sum. The summation of the matrix elements yields back the transfer matrix eigenvalue plus a simple term.

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## Appendix: Symmetric functions of the Bethe roots

Here we derive a number of identities satisfied by symmetric functions of the Bethe roots and symmetric functions of the spectral parameters. As a result we can express any symmetric polynomial that depends on the Bethe roots in terms of symmetric polynomials

of spectral parameters. Let us introduce some notation:

$$\begin{aligned} e_k^B &= e_k(\zeta_1, \dots, \zeta_n), & h_k^B &= h_k(\zeta_1, \dots, \zeta_n), \\ e_k^s &= e_k(z_1, \dots, z_{2n}), & h_k^s &= h_k(z_1, \dots, z_{2n}), \end{aligned}$$

where, as before,  $\zeta$ 's are the Bethe roots and  $z$ 's are the spectral parameters. We will derive this correspondence for both  $e^B$  and  $h^B$ . From this a Schur function of the Bethe roots can be written using the Jacobi-Trudi or the dual Jacobi-Trudi identities (98).

Let us first formulate our general statement. It is based on the proof of the equality of the Jacobi-Trudi with the dual Jacobi-Trudi determinants given e.g. in [12]. Consider three families of parameters:  $a_i$ ,  $b_i$  and  $c_i$ ,  $i \in \mathbb{Z}$ , which satisfy the equations:

$$\sum_{i=0}^k a_i b_{k-i} = c_k, \quad k = 0, 1, 2, \dots \quad (90)$$

$$a_i = b_i = c_i = 0, \quad \text{for } i < 0 \quad (91)$$

Construct three matrices:

$$A = (a_{i-j})_{0 \leq i, j \leq n}, \quad B = (b_{i-j})_{0 \leq i, j \leq n}, \quad \text{and} \quad C = (c_{i-j})_{0 \leq i, j \leq n}. \quad (92)$$

These matrices are lower triangular due to (91). The equations (90) are equivalent to the matrix equation:

$$AB = C, \quad (93)$$

or, if we define a lower triangular matrix  $D = C^{-1}A$ ,  $D = (d_{i-j})_{0 \leq i, j \leq n}$  with entries

$$d_k = (-1)^k \det \begin{pmatrix} \frac{a_0}{c_0} & \frac{a_1}{c_0} & \dots & \frac{a_k}{c_0} \\ 1 & \frac{c_1}{c_0} & \dots & \frac{c_k}{c_0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{c_1}{c_0} \end{pmatrix}, \quad (94)$$

then we can write

$$DB = I. \quad (95)$$

This means that the matrix  $D$  is the inverse of the matrix  $B$ . It follows that each minor of  $D$  is equal to the complementary cofactor of the transposed matrix  $B$ . Recall that for a partition  $\nu = (\nu_1, \dots, \nu_s)$  with  $\nu_1 = p$ , the  $p + s$  numbers:

$$\nu_i + s - i \quad (1 \leq i \leq s), \quad s - 1 + i - \nu'_i \quad (1 \leq i \leq p), \quad (96)$$

form a permutation of  $(0, 1, \dots, p + s - 1)$ . The prime of  $\nu$  is the transposed partition of  $\nu$ . Consider the minor of  $D$  with row indices  $\nu_i + s - i$  ( $1 \leq i \leq s$ ) and column indices  $s - i$  ( $1 \leq i \leq s$ ). This minor is equal up to a sign to the complementary cofactor of the transposed  $B$  which has row indices  $s - 1 + i - \nu'_i$  ( $1 \leq i \leq p$ ) and column indices  $s - 1 + i$  ( $1 \leq i \leq p$ ). We obtain

$$\det_{1 \leq i, j \leq s} d_{\nu_i - i + j} = (-1)^{\sum_{i=1}^s \nu_i} \det_{1 \leq i, j \leq s} b_{\nu'_i - i + j}. \quad (97)$$



Let us look at some application of this formula. If we set  $b_i = (-1)^i e_i$  and  $a_i = h_i$ , where  $e_i$  and  $h_i$  are the elementary symmetric polynomials and the homogeneous symmetric polynomials, respectively, then the conditions (90) and (91) are satisfied with the matrix  $C$  being the identity matrix. Taking this into account in (94), we get  $d_k = h_k$ , and (97) reads

$$s_\nu = \det_{1 \leq i, j \leq s} h_{\nu_i - i + j} = \det_{1 \leq i, j \leq s} e_{\nu'_i - i + j}, \quad (98)$$

This is the Jacobi-Trudi and the dual Jacobi-Trudi identities for the Schur function  $s_\nu$ .

Now we will use the  $T - Q$  relation (52) to derive equations of the form (90) for various symmetric polynomials. Let us first recall the expression from the Appendix of the paper [7]

$$F(q^2 t) Q(q^2 t) = \frac{\begin{pmatrix} z_1^{3n} & z_2^{3n} & \dots & z_{2n}^{3n} & t^{3n} \\ \vdots & \vdots & & \vdots & \vdots \\ z_1^{3k+2} & z_2^{3k+2} & \dots & z_{3n}^{3k+2} & t^{3k+2} \\ z_1^{3k} & z_2^{3k} & \dots & z_{3n}^{3k} & t^{3k} \\ \vdots & \vdots & & \vdots & \vdots \\ z_1^2 & z_2^2 & \dots & z_{3n}^2 & t^2 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}}{\begin{pmatrix} z_1^{3n-1} & z_2^{3n-1} & \dots & z_{2n}^{3n-1} \\ \vdots & \vdots & & \vdots \\ z_1^{3k+2} & z_2^{3k+2} & \dots & z_{3n}^{3k+2} \\ z_1^{3k} & z_2^{3k} & \dots & z_{3n}^{3k} \\ \vdots & \vdots & & \vdots \\ z_1^2 & z_2^2 & \dots & \\ 1 & 1 & \dots & 1 \end{pmatrix}}, \quad (99)$$

with  $Y'_{2n} = \{n, n, \dots, 1, 1, 0\}$ . This expression gives the  $Q$  function and hence all coefficients of the expansion of the  $Q(q^2 t)$  in powers of  $t$ . These coefficients are the elementary symmetric polynomial of the Bethe roots:

$$Q(t) = q^{2n} \sum_{i=0}^n (-1)^i t^{n-i} q^i e_i^B. \quad (100)$$

We would like to redefine the  $e^B$  in order to avoid carrying around the factors of  $q$ . Let  $\tilde{e}_i^B = q^i e_i^B$ , and we omit the tilde in this appendix by abuse of notation. The function  $F$  in (99) is the generating function of the elementary symmetric polynomials in the spectral parameters:

$$F(t) = q^{4n} \sum_{i=0}^{2n} (-1)^i t^{2n-i} e_i^s. \quad (101)$$

Now we expand both sides of the equation (99) in powers of  $t$ :

$$\sum_{k=0}^{3n} (-1)^k t^{3n-k} \sum_{j=0}^k e_j^B e_{k-j}^s = \sum_{k=0}^{3n} (-1)^k t^{3n-k} \gamma_k. \quad (102)$$

The coefficients  $\gamma_k$  are Schur functions which are labeled by partitions  $\pi_k$  divided by the Schur function  $s_{\tilde{Y}_n}$ . These Schur functions can be obtained from the expansion in  $t$  of the Schur function  $s_{Y'_{2n+1}}$  in (99). The partitions  $\pi_k$  are derived from  $Y'$  in the following way. The partition  $Y'_{2n+1}$  has length equal to  $2n+1$ , and we can derive  $2n+1$  other partitions as follows:

$$U^{(j)} = Y' + \theta^{(j)}, \quad (103)$$

$$\theta_i^{(j)} = \begin{cases} 0 & \text{if } i \geq j \\ 1 & \text{otherwise.} \end{cases}$$

The partitions  $\pi_k$  are written in terms of  $U^{(k)}$  as:

$$\pi_{k-1} = \{U_1^{(k)}, \dots, \hat{U}_k^{(k)}, \dots, U_{2n+1}^{(k)}\}, \quad (104)$$

so  $\pi_k$  is the  $U^{(k)}$  with the  $k+1$ 'st part absent ( $k = 0, \dots, 2n$ ). The expansion of the Schur function  $s'_{Y'_{2n+1}}$  in (99) gives  $3n+1$  coefficients  $\gamma_i$  some of which are equal to zero, others are Schur functions of  $\pi_k$ :

$$\gamma_{3j} = (-1)^j \frac{s^{\pi_{2j}}}{s_{\tilde{Y}}}, \quad j = 0, \dots, n$$

$$\gamma_{3j+1} = (-1)^j \frac{s^{\pi_{2j+1}}}{s_{\tilde{Y}}}, \quad \gamma_{3j+2} = 0, \quad j = 0, \dots, n-1. \quad (105)$$

Let us get back to (102). Collecting the coefficients over  $t$  we get:

$$\sum_{j=0}^k e_j^B e_{k-j}^s = \gamma_k. \quad (106)$$

Setting  $a$  to  $e^s$ ,  $b$  to  $e^B$  and  $c$  to  $\gamma$  (note that  $\gamma_0 = 1$ ), and write in this particular case  $f$  instead of  $d$ , we obtain:

$$\det_{1 \leq i, j \leq s} f_{\nu_i - i + j} = \det_{1 \leq i, j \leq s} e_{\nu'_i - i + j}^B. \quad (107)$$

with

$$f_k = (-1)^k \det \begin{pmatrix} 1 & e_1^s & \dots & e_k^s \\ 1 & \gamma_1 & \dots & \gamma_k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_1 \end{pmatrix}, \quad (108)$$

which effectively means that  $h_k^B = f_k$ . The polynomials  $f_k$  can also be written as a sum:

$$f_k = \sum_{j=0}^k (-1)^j e_{k-j} \delta_j, \quad (109)$$

where  $\delta_j$  are the entries of the inverse of  $C$  with  $c_i = \gamma_i$  which can be written as

$$\delta_k = \det_{1 \leq i, j \leq k} \gamma_{1-i+j}. \quad (110)$$

We also can write  $\gamma_k$  in terms of  $\delta_i$ :

$$\gamma_k = \det_{1 \leq i, j \leq k} \delta_{1-i+j}. \quad (111)$$

Since  $\gamma$  and  $\delta$  are dual we can write the equations (90) for them:

$$\sum_{i=0}^k (-1)^i \gamma_i \delta_{k-i} = 0, \quad k = 0, 1.. \quad (112)$$

Now if we take a sum of  $\gamma$  with the  $f$ 's the previous equation together with the eq.(109) allow to obtain the polynomials  $e^s$ :

$$\sum_{i=0}^k \gamma_i f_{k-i} = e_k^s, \quad k = 0, 1.., 2n. \quad (113)$$

Since  $f_i = h_i^B$ , we get another equation for a determinat of the Bethe roots:

$$\det_{1 \leq i, j \leq s} g_{\nu'_i - i + j} = (-1)^{\sum_j \nu_j} \det_{1 \leq i, j \leq s} h_{\nu_i - i + j}^B, \quad (114)$$

where  $g_i$  can be written as a determinant:

$$g_k = \det \begin{pmatrix} 1 & \gamma_1^s & \dots & \gamma_k^s \\ 1 & e_1^s & \dots & e_k^s \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e_1^s \end{pmatrix}, \quad (115)$$

and also as a sum

$$g_k = \sum_{j=0}^k (-1)^k h_{k-j}^s \gamma_j. \quad (116)$$

It follows also that  $e_i^B = g_i$ . Let us summarize what we obtained:

$$e_k^s = \sum_{i=0}^k \gamma_{k-i} h_i^B, \quad h_k^s = \sum_{i=0}^k (-1)^i \delta_{k-i} e_i^B, \quad (117)$$

$$e_k^B = \sum_{i=0}^k \gamma_{k-i} h_i^s, \quad h_k^B = \sum_{i=0}^k (-1)^{i+k} \delta_{k-i} e_i^s. \quad (118)$$

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